Multivariate Calculus Solution[1]

Math Camp 2012

Hessian Matrices

In 1st semester micro, you will solve general equilibrium models. Sometimes when solving these models it is useful to see if utility functions are concave. One way of testing for concavity involves calculating the function's Hessian. Find the Hessian matrices of the following utility functions (these functions were used in previous homeworks and tests):

1. (Core Exam) $U(x_1, x_2) = \frac{2}{3}\sqrt{x_1} + \frac{1}{3}\sqrt{x_2}$

$$\frac{\partial u}{\partial x_1} = \frac{1}{3}x_1^{-\frac{1}{2}} \quad \frac{\partial^2 u}{\partial x_1^2} = -\frac{1}{6}x_1^{-\frac{3}{2}} \quad \frac{\partial^2 u}{\partial x_2 \partial x_1} = 0$$
$$\frac{\partial u}{\partial x_2} = \frac{1}{6}x_2^{-\frac{1}{2}} \quad \frac{\partial^2 u}{\partial x_2^2} = -\frac{1}{12}x_2^{-\frac{3}{2}} \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$$

Therefore

$$H = \begin{bmatrix} -\frac{1}{6}x_1^{-\frac{3}{2}} & 0 \\ 0 & -\frac{1}{12}x_2^{-\frac{3}{2}} \end{bmatrix}$$

2. (Final Exam) $U(x_1, x_2) = x_1 + \frac{\delta}{\alpha}x_2^{\alpha}$

$$\frac{\partial u}{\partial x_1} = 1 \quad \frac{\partial^2 u}{\partial x_1^2} = 0 \quad \frac{\partial^2 u}{\partial x_2 \partial x_1} = 0$$
$$\frac{\partial u}{\partial x_2} = \delta x_2^{\alpha-1} \quad \frac{\partial^2 u}{\partial x_2^2} = (\alpha - 1)\delta x_2^{\alpha-2} \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$$

Therefore

$$H = \begin{bmatrix} 0 & 0 \\ 0 & (\alpha - 1)\delta x_2^{\alpha-2} \end{bmatrix}$$

3. (Homework problem) $U(x_1, x_2, x_3) = -\frac{1}{x_1} + x_2 - \delta\frac{1}{x_3}$

$$\frac{\partial u}{\partial x_1} = x_1^{-2} \quad \frac{\partial^2 u}{\partial x_2 \partial x_1} = \frac{\partial^2 u}{\partial x_3 \partial x_1} = 0 \quad \frac{\partial^2 u}{\partial x_1^2} = -2x_1^{-3}$$
$$\frac{\partial u}{\partial x_2} = 1 \quad \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 u}{\partial x_3 \partial x_2} = 0$$
$$\frac{\partial u}{\partial x_3} = \delta x_3^{-2} \quad \frac{\partial^2 u}{\partial x_3^2} = -2\delta x_3^{-3} \quad \frac{\partial^2 u}{\partial x_1 \partial x_3} = \frac{\partial^2 u}{\partial x_2 \partial x_3} = 0$$

Therefore

$$H = \begin{bmatrix} -2x_1^{-3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2\delta x_3^{-3} \end{bmatrix}$$

[1] If you find any typo please email me: Maria_Jose_Boccardi@Brown.edu

4. (Midterm Exam) $U(x_1, x_2) = \frac{1}{3}\ln(x_1) + \frac{2}{3}\ln(x_2)$

$$\frac{\partial u}{\partial x_1} = \frac{1}{3x_1} \quad \frac{\partial^2 u}{\partial x_1^2} = -\frac{1}{3x_1^2} \quad \frac{\partial^2 u}{\partial x_2 \partial x_1} = 0$$
$$\frac{\partial u}{\partial x_2} = \frac{2}{3x_2} \quad \frac{\partial^2 u}{\partial x_2^2} = -\frac{2}{3x_2^2}$$

Therefore

$$H = \begin{bmatrix} -\frac{1}{3x_1^2} & 0 \\ 0 & -\frac{2}{3x_2^2} \end{bmatrix}$$

Slutsky equation

The Slutsky Equation breaks changes in demand into income effects and substitution effects. Last semester for one of the homework problems, we were asked to calculate the Slutsky equation to find the substitution effect and the income effect. In the problem, the utility function was the same as in question 3 above, and it can be shown that the direct demand functions are

$$x_1(p_1, p_2, w) = \frac{w}{3p_1}$$

$$x_2(p_1, p_2, w) = \frac{2w}{3p_2}$$

Let the function $\mathbf{x}(p, w) : \mathbb{R}^3 \to \mathbb{R}^2$ be the combined demand function, where $p = (p_1, p_2)$. The Slutsky equation is as follows:

$$D_p\mathbf{x} + D_w\mathbf{x} \cdot \mathbf{x}' = D_p\mathbf{v},$$

where the subscripts denote which derivative it is respect to (Note: these are total derivatives, not directional derivatives) and v is the indirect utility function. Find $D_p\mathbf{x}$ (the total effect), $D_w\mathbf{x} \cdot \mathbf{x}'$ (the wealth effect), and use the above Slutsky equation to compute $D_p\mathbf{v}$ (the substitution effect). Hint: $D_p\mathbf{x}$ is a 2×2 matrix, $D_w\mathbf{x}$ is 1×2, and $\mathbf{x}$ is 1×2. This involves matrix multiplication, something we haven't covered yet, but that you should already know.

$$x(p, w) = \begin{pmatrix} \frac{w}{3p_1} \\ \frac{2w}{3p_2} \end{pmatrix} == \begin{pmatrix} x_1(p, w) \\ x_2(p, w) \end{pmatrix}$$

Therefore

$$x' = \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right)$$

Then we have that

$$D_p x = \begin{pmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{pmatrix} = \begin{pmatrix} -\frac{w}{3p_1^2} & 0 \\ 0 & -\frac{2w}{3p_2^2} \end{pmatrix}$$

$$D_w x = \begin{pmatrix} \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial w} \end{pmatrix} = \begin{pmatrix} \frac{1}{3p_1} \\ \frac{2}{3p_2} \end{pmatrix}$$

$$D_w x \cdot x' = \begin{pmatrix} \frac{1}{3p_1} \\ \frac{2}{3p_2} \end{pmatrix} \left(\frac{w}{3p_1}, \frac{2w}{3p_2}\right) = \begin{pmatrix} \frac{w}{9p_1^2} & \frac{2w}{9p_1 p_2} \\ \frac{2w}{9p_1 p_2} & \frac{4w}{9w_2^2} \end{pmatrix}$$

$$D_p x + D_w x \cdot x' = \begin{pmatrix} -\frac{2w}{9p_1^2} & \frac{2w}{9p_1 p_2} \\ \frac{2w}{9p_1 p_2} & -\frac{2w}{9p_2^2} \end{pmatrix} = D_p v$$

Gradient

If the graph of a function $F : \mathbb{R}^3 \to \mathbb{R}^1$ lives in $\mathbb{R}^4$, in which space does the gradient ∇F live?

The gradient lives in $\mathbb{R}^3$

Integration

Evaluate the following integrals:

1. $\int_0^2 \int_0^{4-x^2} xydydx$

$$\begin{aligned}\int_0^2 \int_0^{4-x^2} xydydx &= \int_0^2 \left(\frac{1}{2}xy^2|_0^{4-x^2}\right) dx \\ &= \int_0^2 \left(\frac{1}{2}x(4-x^2)^2\right) dx \\ &= \int_0^2 \left(8x - 4x^3 + \frac{1}{2}x^5\right) dx \\ &= \left[4x^2 - x^4 + \frac{1}{12}x^6\right]_0^2 = \frac{16}{3}\end{aligned}$$

2. $\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} x^2ydydx$

$$\begin{aligned}\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} x^2ydydx &= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{1-x}^{\sqrt{1-x^2}} dx \\ &= \int_0^1 \frac{1}{2}\left[x^2(1-x^2) - x^2(1-x)^2\right] dx \\ &= \int_0^1 \left[x^3 - x^4\right] dx \\ &= \left[\frac{1}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{20}\end{aligned}$$

3. $\int\int xdydx$ for the region bounded by $y = x$ and $y = 3 - x^2$

   In figure 3 we can see the integration region, where the two curves intersect at $x = \frac{1-\sqrt{13}}{2}$ and $x = -\frac{1-\sqrt{13}}{2}$, therefore we can set the integral as

$$\int_{-\frac{1-\sqrt{13}}{2}}^{-\frac{1-\sqrt{13}}{2}} \int_x^{3-x^2} xdydx$$

   Therefore we have that

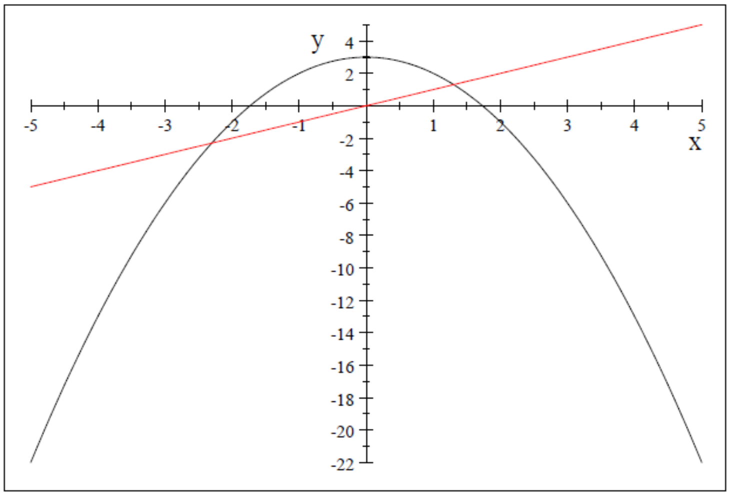

Figure 1: Region

$$\begin{aligned}
\int_{\frac{1-\sqrt{13}}{2}}^{-\frac{1-\sqrt{13}}{2}} \int_{x}^{3-x^2} xdydx &= \int_{\frac{1-\sqrt{13}}{2}}^{-\frac{1-\sqrt{13}}{2}} [xy]_x^{3-x^2}\, dx \\
&= \int_{\frac{1-\sqrt{13}}{2}}^{-\frac{1-\sqrt{13}}{2}} \left[x(3-x^2) - x^2\right] dx \\
&= \int_{\frac{1-\sqrt{13}}{2}}^{-\frac{1-\sqrt{13}}{2}} \left[3x - x^3 - x^2\right] dx \\
&= \left[\frac{3}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{3}x^3\right]_{\frac{1-\sqrt{13}}{2}}^{-\frac{1-\sqrt{13}}{2}} \\
&= \left[\frac{3}{2}\left(-\frac{1-\sqrt{13}}{2}\right)^2 - \frac{1}{4}\left(-\frac{1-\sqrt{13}}{2}\right)^4 - \frac{1}{3}\left(-\frac{1-\sqrt{13}}{2}\right)^3\right] - \left[\frac{3}{2}\left(\frac{1-\sqrt{13}}{2}\right)^2 - \frac{1}{4}\left(1 -\right.\right.
\end{aligned}$$

4. $\int_0^1 \int_y^1 x^2 \sin(xy) dxdy$
Let's first exchange the order of integration, see graph 4, therefore we have that

$$\int_0^1 \int_y^1 x^2 \sin(xy) dxdy = \int_0^1 \int_0^x x^2 \sin(xy) dydx$$

Let's focus first in the inner integral, then we have that

$$\begin{aligned}
\int_0^x x^2 \sin(xy) dy &= x^2 \int_0^x \sin(xy) dy \\
&= x^2 \left[-\frac{1}{x}\cos(xy)\right]_0^x \\
&= x\left[-\cos(x^2)\right] + x\left[\cos(0)\right] \\
&= x\left[1 - \cos(x^2)\right] = x - x\cos(x^2)
\end{aligned}$$

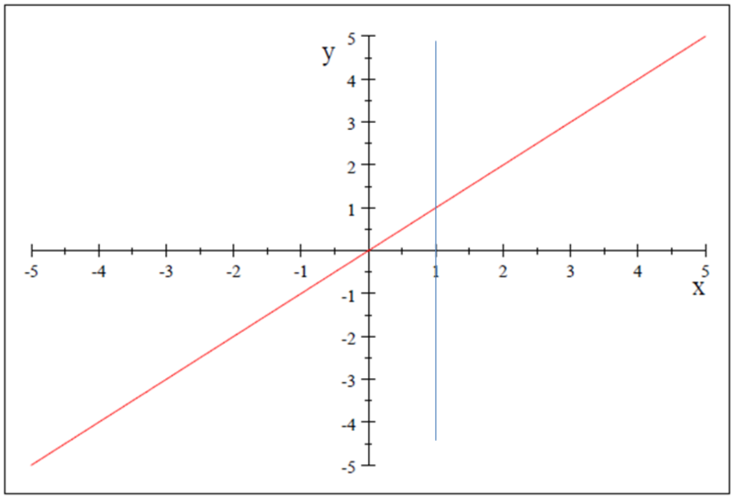

Figure 2: Region

Now going back to the other integral and using substitution we have that

$$\begin{aligned}
\int_0^1 \int_0^x x^2 \sin(xy) dy dx &= \int_0^1 \left[x - x\cos(x^2)\right] dx \\
&= \int_0^1 x dx - \int_0^1 x\cos(x^2) dx \\
&= \left[\frac{x^2}{2}\right]_0^1 - \int_0^1 x\cos(x^2) dx \\
&= \frac{1}{2} - \int_0^1 x\cos(x^2) dx \\
f(x) = \cos x \quad u = g(x) = x^2 \Rightarrow & \ du = 2x dx \\
&= \frac{1}{2} + \int_{g(0)}^{g(1)} -\frac{1}{2}\cos u du \\
&= \frac{1}{2} - \frac{1}{2}\left[\sin u\right]_0^1 = +\frac{1}{2} - \frac{1}{2} sin 1
\end{aligned}$$

Sketch the following regions:

1. $1 \le x \le 2,\ 2 \le y \le 7$

   See figure 1

2. $0 \le x \le 2,\ \frac{x^2}{2} \le y \le x$

   See figure 2

3. $1 \le x \le 2,\ x^2 \le y \le x+2$

   See figure 3

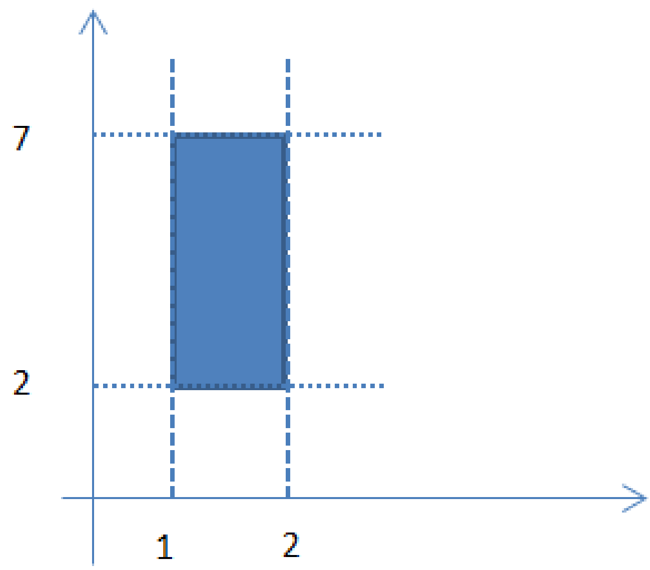

Figure 3: Region

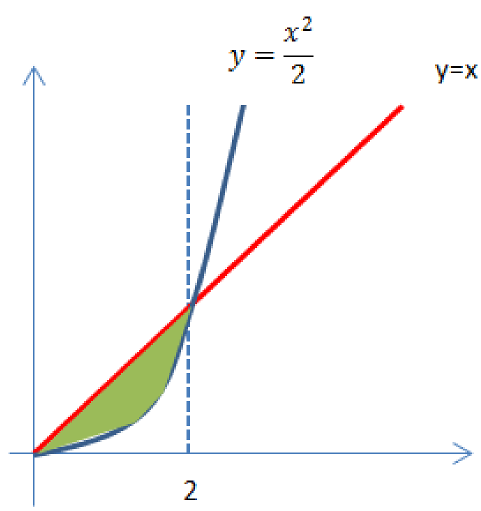

Figure 4: Region

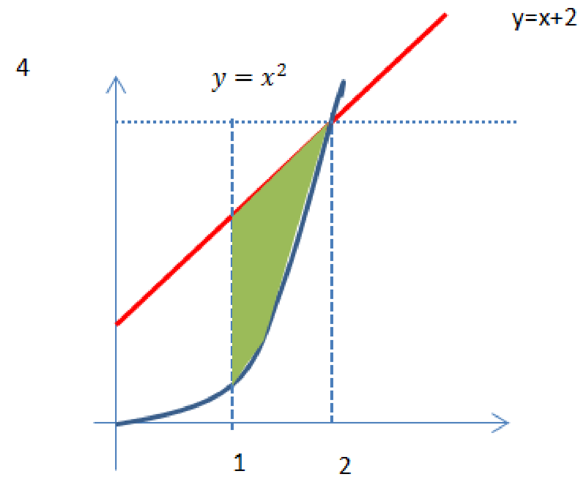

Figure 5: Region

4. $0 \leq x \leq 1$, $x^2 \leq y \leq x$

   See figure 4

Integrate the following:

See figure for the integration region

1. $f(x, y) = \sin(x^2)$ over the area $0 \leq x \leq 2$, $0 \leq y \leq \frac{x}{2}$ with respect to x first

   Therefore we can rewrite the integral as

$$\int_0^1 \int_{2y}^2 \sin(x^2) dx dy$$

   There is not an easy way to solve it, see http:en.wikipedia.orgwikiFresnel_integral

2. $f(x, y) = \sin(x^2)$ over the area $0 \leq x \leq 2$, $0 \leq y \leq \frac{x}{2}$ with respect to y first
   Therefore we can rewrite the integral as

$$\int_0^2 \int_0^{\frac{x}{2}} \sin(x^2) dy dx$$

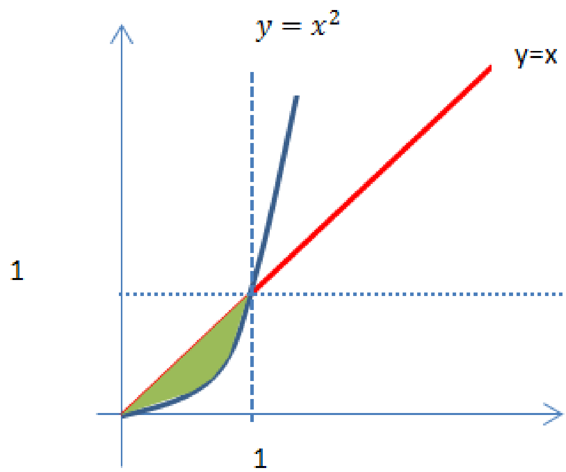

Figure 6: Region

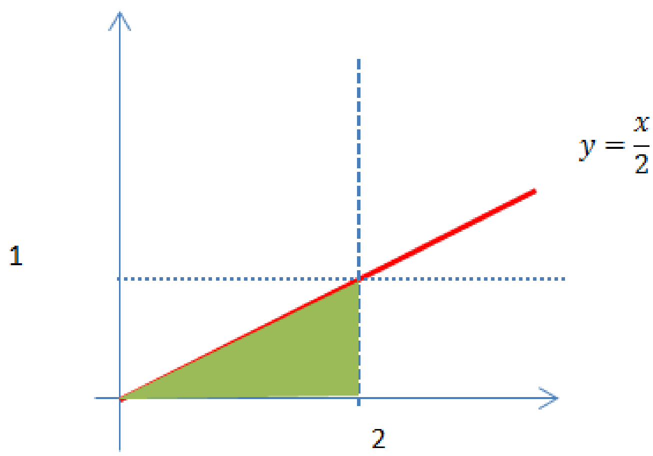

Figure 7: Region

and we have that

$$\begin{aligned}
\int_0^2 \int_0^{\frac{x}{2}} \sin(x^2) dy dx &= \int_0^2 \left[\sin(x^2) y\right]_0^{\frac{x}{2}} dx \\
&= \int_0^2 \left[\frac{x}{2}\sin(x^2)\right] dx \\
f(x) = \frac{1}{4}\sin(x) \quad u = g(x) = x^2 &\Rightarrow g'(x) = 2x \Rightarrow du = 2x dx \\
&= \int_0^4 \left[\frac{1}{4}\sin(u)\right] du \\
&= -\left[\frac{1}{4}\cos(u)\right]_0^4 = \frac{1}{4}\left[1 - \cos(4)\right]
\end{aligned}$$

Differentiate the following:

1. $\int_x^1 \lambda e^{-\lambda x y} dy$ with respect to x.
   We need to use Leibnitz's Integral rule. Remind that

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y) dx = \int_{a(y)}^{b(y)} \frac{\partial f}{\partial y} dx + f[b(y), y]\frac{db}{dy} - f[a(y), y]\frac{da}{dy}$$

which can be rewritten as

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y) dy = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x} dy + f[x, b(x)]\frac{db}{dx} - f[x, a(x)]\frac{da}{dx}$$

Therefore we have that

$$
\begin{aligned}
\frac{d}{dx}\int_x^1 \lambda e^{-\lambda xy}dy &= \int_x^1 \frac{\partial \lambda e^{-\lambda xy}}{\partial x}dy - f[x,x] \\
&= \int_x^1 -\lambda y \lambda e^{-\lambda xy}dy - \lambda e^{-\lambda x^2} \\
&= \int_x^1 -\lambda^2 y e^{-\lambda xy}dy - \lambda e^{-\lambda x^2} \\
&= -\lambda^2 \int_x^1 y e^{-\lambda xy}dy - \lambda e^{-\lambda x^2} \\
f(y) = y \quad g'(y) = e^{-\lambda xy} &\Rightarrow g(y) = \frac{e^{-\lambda xy}}{-\lambda x} \\
&= -\lambda^2 \left[\left[y\frac{e^{-\lambda xy}}{-\lambda x}\right]_x^1 - \int_x^1 \frac{e^{-\lambda xy}}{-\lambda x}dy\right] - \lambda e^{-\lambda x^2} \\
&= \lambda \left[\left[\frac{ye^{-\lambda xy}}{x}\right]_x^1 - \int_x^1 e^{-\lambda xy}dy\right] - \lambda e^{-\lambda x^2} \\
&= \lambda \left[\left[\frac{e^{-\lambda x} - xe^{-\lambda x^2}}{x}\right] - \int_x^1 e^{-\lambda xy}dy\right] - \lambda e^{-\lambda x^2} \\
&= \lambda \left[e^{-\lambda x}\left[\frac{1 - xe^{x}}{x}\right] - \int_x^1 e^{-\lambda xy}dy\right] - \lambda e^{-\lambda x^2} \\
&= \lambda \left[e^{-\lambda x}\left[\frac{1 - xe^{x}}{x}\right] - \left[\frac{e^{-\lambda xy}}{-\lambda x}\right]_x^1\right] - \lambda e^{-\lambda x^2} \\
&= e^{-\lambda x}\left[1 + \lambda\left(\left[\frac{1}{x} - 2e^{x}\right]\right) - \frac{e^{x}}{x}\right]
\end{aligned}
$$

2. $\int_x^{e^x} 2xe^{xy}dy$ with respect to x.

$$
\begin{aligned}
\frac{d}{dx}\int_x^{e^x} 2xe^{xy}dy &= \int_x^{e^x} \frac{\partial f}{\partial x}dy + f[x,e^x]\frac{de^x}{dx} - f[x,x]\frac{dx}{dx} \\
&= \int_x^{e^x} \left[2e^{xy} + 2xye^{xy}\right]dy + 2xe^{xe^x}e^x - 2xe^{x^2}
\end{aligned}
$$

Working with pdf In 1st semester econometrics, you will be asked to integrate probability density functions (or p.d.f.s) in order to find the probability that certain events will occur. Sometimes it is necessary to transform the random variables. For example, if we are given the density functions for X_1 and X_2, and Y_1 and Y_2 as functions of X_1 and X_2, we can then transform this system to solve for the p.d.f.'s of Y_1 and Y_2. The following problems are taken from p.d.f.'s in the first semester econometrics textbook, section 3.7. For each of the following,

- Find X_1 and X_2 as functions of Y_1 and Y_2
- Find the determinant of the Jacobian
- Sketch the region of integration in terms of X_1 and X_2
- Sketch the region of integration in terms of Y_1 and Y_2
- Evaluate the new integral in terms of Y_1 and Y_2

1. $f(X_1, X_2) = e^{-X_1 - X_2}$ over the region $X_1 > 0$, $X_2 > 0$, where $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$. Since we have that $Y_1 = \frac{X_1}{X_1+X_2}$ and $Y_2 = X_1 + X_2$, we can solve for X_1 and X_2 and obtain that $X_1 = Y_1 Y_2$ and $X_2 = Y_2(1 - Y_1)$. Computing the Jacobian we have that

$$\tfrac{\partial X_1}{\partial Y_1} = Y_2 \quad \tfrac{\partial X_1}{\partial Y_2} = Y_1 \quad \tfrac{\partial X_2}{\partial Y_1} = -Y_2 \quad \tfrac{\partial X_2}{\partial Y_2} = (1 - Y_1)$$

therefore

$$|J| = |\frac{\partial X_1}{\partial Y_1}\frac{\partial X_2}{\partial Y_2} - \frac{\partial X_1}{\partial Y_2}\frac{\partial X_2}{\partial Y_1}| = |Y_2(1 - Y_1) + Y_1 Y_2| = |Y_2|$$

The original integration region was given by $X_1 > 0$ and $X_2 > 0$, therefore we have that $Y_1 Y_2 > 0$ and $Y_2(1 - Y_1) > 0$ which is possible if and only if $Y_2 > 0$ and $0 < Y_1 < Y_2$.

Therefore we can rewrite the integration as

$$\begin{aligned}
\int_0^\infty \int_0^\infty e^{-X_1 - X_2} dX_1 dX_2 &= \int_0^\infty \int_0^1 e^{-Y_1Y_2 - (1-Y_1)Y_2} |Y_2| dY_1 dY_2 \\
&= \int_0^\infty \int_0^1 e^{-Y_2} Y_2 dY_1 dY_2 \\
&= \int_0^\infty \left[e^{-Y_2} Y_2 Y_1\right]_0^1 dY_2 \\
&= \int_0^\infty e^{-Y_2} Y_2 dY_2 \\
&= -e^{Y_2} Y_2|_0^\infty + \int_0^\infty e^{-Y_2} dY_2 \\
&= 0 - e^{-Y_2}|_0^\infty = -e^{-\infty} + e^0 = 1
\end{aligned}$$

2. $f(X_1, X_2) = 8X_1X_2$ over the region $0 \le X_1 \le X_2 \le 1$, where $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$. Since we have that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$, we can solve for X_1 and X_2 and obtain that $X_1 = Y_1 Y_2$ and $X_2 = Y_2$. Computing the Jacobian we have that

$$\tfrac{\partial X_1}{\partial Y_1} = Y_2 \quad \tfrac{\partial X_1}{\partial Y_2} = Y_1 \quad \tfrac{\partial X_2}{\partial Y_1} = 0 \quad \tfrac{\partial X_2}{\partial Y_2} = 1$$

therefore

$$|J| = |\frac{\partial X_1}{\partial Y_1}\frac{\partial X_2}{\partial Y_2} - \frac{\partial X_1}{\partial Y_2}\frac{\partial X_2}{\partial Y_1}| = |Y_2|$$

The original integration region was given by $0 \le X_1 \ge X_2 \le 1$, therefore we have that $0 \le Y_1 Y_2 \ge Y_2 \le 1$ which is possible if and only if $0 \le Y_1 \le 1$ and $0 \le Y_2 \le 1$.

Therefore we can rewrite the integration as

$$\begin{aligned}
\int_0^1 \int_0^{X_2} 8X_1X_2 dX_1 dX_2 &= \int_0^1 \int_0^1 8Y_1Y_2Y_2|Y_2| dY_1 dY_2 \\
\int_0^1 \int_0^1 8Y_1 Y_2^3 dY_1 dY_2 \\
\int_0^1 [4Y_1^2 Y_2^3]_0^1 dY_2 \\
\int_0^1 [4Y_2^3] dY_2 \\
Y_2^4|_0^1 = 1
\end{aligned}$$

Hint: As a check, realize that all p.d.f.'s integrate to one. So the original integrals with respect to X_1 and X_2, as well as the transformed integral with respect to Y_1 and Y_2, should integrate to one. Try it!